λ - calculus

lecture 1

FP-Bremen

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Λ - calculus

application, abstraction

$\Lambda ::= v \mid \Lambda\ \Lambda \mid \lambda v.\Lambda$

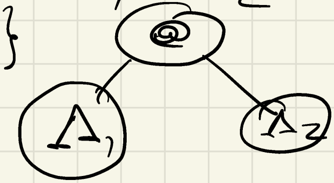
right ass

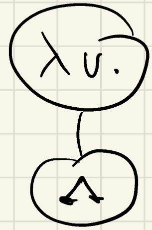
left

$\Lambda_1 @ \Lambda_2$

v

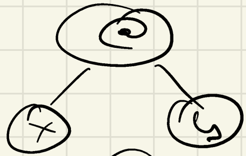
$v \in V = \{x, y, \ldots\}$





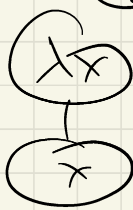
x

$x\ y \leftarrow$



$x, y \in V$

$\lambda x.x \leftarrow$



$x \in V$

$\lambda x.\ \text{body} \equiv \backslash x \to \text{body}$

$f\ x = \text{body}$

$\lambda z.(\lambda x.x\,y)\,z$

$\lambda xyz.e \equiv$

$\equiv \lambda x.(\lambda y.(\lambda z.e))$

$abcd \equiv ((ab)c)d$

$\lambda x.\lambda y.e \equiv \lambda x.(\lambda y.e)$

$(\lambda x.e_1)\,e_2 \qquad \lambda x.(e_1\,e_2)$

def $FV(T)$ ← free variables

$$FV(x) = \{x\}$$
$$FV(e_1 e_2) = FV(e_1) \cup FV(e_2)$$
$$FV(\lambda x.e) = FV(e) \setminus \{x\}$$

$BV(T)$ ← bound variables

$$BV(x) = \emptyset$$
$$BV(e_1 e_2) = BV(e_1) \cup BV(e_2)$$
$$BV(\lambda x.e) = BV(e) \cup \{x\}$$

bound, free

$$T = \lambda x.\, x\; y$$

$$BV(T) = \{x\} \qquad FV(T) = \{y\}$$

def T - closed (kombinator) if $FV(T) = \emptyset$

ex:

$$I = \lambda x.x$$
$$S = \lambda f.\lambda g.\lambda x.\, f\, x\, (g\, x)$$
$$K = \lambda x.\lambda y.\, x$$
$$K_* = \lambda x.\lambda y.\, y$$

bound, free

$$(\lambda x.x)\; x$$

$$(\lambda y.y)\; x$$

Barendregt's conversion

α - conversion

$$\lambda x.x \sim_\alpha \lambda y.y$$

β-reduction

$$\underbrace{(\lambda x. e_1) e_2}_{\text{redex}} \to_\beta \underbrace{e_1 [x / e_2]}_{\text{в } e_1 \text{ всяко вхожд. } x \text{, } e_2}$$

ex $$(\lambda x. \overbrace{x\ x}^{e_1}) \underbrace{(\lambda f. \lambda s. s)}_{e_2}$$

$$\to_\beta (\lambda f. \lambda s. s)(\lambda f. \lambda s. s)$$

$$\underbrace{(\lambda x. x)(\underbrace{(\lambda y. y) z}_{\text{redex}})}_{\text{redex}}$$

$$(\lambda y . \lambda x . x y) x \not\to_\beta \lambda x . x x$$

bound

free

$$\downarrow \sim_\alpha$$

$$(\lambda y . \lambda z . z y) x \to_\beta \lambda z . z x$$

$$(\lambda y . \underline{\lambda x_1 . x_1 y}) x_2$$

Capture-avoiding substitution

Static binding

def

(1) $x[x/M] = M$

(2) $y[x/M] = y\ ,\quad x \neq y \quad (\in V)$

(3) $(e_1 e_2)[x/M] = (e_1[x/M])(e_2[x/M])$

(4) $(\lambda x. e)[x/M] = \lambda x. e$ if $e_1 \neq \lambda x. \ldots$

$\rightarrow (\lambda y. e)[x/M] = \lambda y. e[x/M]$ if $x \neq y \wedge y \notin FV(M)$

$\rightarrow (\lambda y. e)[x/M] = \lambda z. e[y/z][x/M]$ if z - fresh and $y \in FV(M)$

$$\overbrace{(\lambda y. \underbrace{\lambda x. x y}_{e_1})}^{\text{redex}}(\underset{e_2}{x}) \rightarrow_\beta$$

$$\rightarrow_\beta (\lambda x. x y)[y/x]$$

$$= \lambda z. \underbrace{(x y)[\cancel{x}/z]}_{\overset{(3)}{=}}[y/x]$$

$$\underbrace{x[x/z]}_{\overset{(1)}{=}} \quad \underbrace{y[x/z]}_{\overset{(2)}{=}}$$

$$z \qquad\qquad y$$

$$= \lambda z.\underbrace{(z\,y)[y/x]}_{\overset{(3)}{=}}$$

$$z[y/x] \quad y[y/x]$$

$$\overset{(2)}{=} \qquad \overset{(1)}{=}$$

$$z \qquad x$$

$$= \lambda z.z\,x$$

$$((x\,y)(x\,x))[x/M]$$

$$(M\,y)(M\,M)$$

$$(\lambda x.\,x\,abc\;x\,\ldots)[x/M]$$

$$= \lambda x.\,x\,abc\;x\ldots$$

$$(\lambda x.\lambda y.\,x\,y)(\lambda z.z) \to_\beta$$

$$\to_\beta \lambda y.(\lambda z.z)\,y \to_\beta \lambda y.y$$

$$\cancel{(\lambda.\lambda.2\,1)(\lambda.1) \to_\beta \lambda.(\lambda.1)\,1 \to \lambda.1}$$

$$(\lambda.\lambda.1\,0)(\lambda.0) \to_\beta \lambda.(\lambda.0)\,0 \to_\beta \lambda.0$$

de Bruijn

NB! Считаем с $\emptyset$!!

$$\Lambda^d ::= \mathbb{N}^{\nearrow k} \mid \Lambda^d\,\Lambda^d \mid \lambda.\Lambda^d$$

$$S = \lambda f.\lambda g.\lambda x.\, f\, x\, (g\, x)$$

no redexes

$$= \lambda.\lambda.\lambda.\, 3\, 1\, (2\, 1)$$

$$\lambda.\lambda.\lambda.\, 2\, 0\, (1\, 0)$$

Subst

$$k[j/M] = \begin{cases} M, & k=j \\ k, & \text{otherwise} \end{cases}$$

$$(e_1 e_2)[j/M] = (e_1[j/M])(e_2[j/M])$$

$$(\lambda.e)[j/M] = \lambda.\, e[j+1/\uparrow^1_0 M]$$

Shift

$$\uparrow^d_c(k) = \begin{cases} k, & k<c \\ k+d, & k \geq c \end{cases}$$

$$\uparrow^d_c(e_1 e_2) = \uparrow^d_c(e_1)\ \uparrow^d_c(e_2)$$

$$\uparrow^d_c(\lambda.e) = \lambda.\uparrow^d_{c+1} e$$

β-reduction

$$(\lambda.e_1)e_2 \to_\beta \uparrow^{-1}_0\left(e_1[0/\uparrow^1_0(e_2)]\right)$$

$$\lambda x.\ \underbrace{(\lambda y.\ \lambda z.\ y z)\ \underbrace{(\lambda m.\ m x)}}$$

$$\to_\beta \lambda x.\ \lambda z.\ (\lambda m.\ m x)\ z \to_\beta \ldots$$

$$\to_\beta \lambda x. \lambda z.\ z\ x$$

$$\cancel{\lambda.\ \underbrace{(\lambda.\ \lambda.\ 2\ 1)\ (\lambda.\ 1\ 2)}}$$

$$\cancel{\to_\beta \lambda.\ (\lambda.\ 2\ 1)\ [\mathbf{1} / \lambda.\ 1 2]}$$

$$\cancel{= \lambda.\ \lambda.\ (2\ 1)\ [2 / \underbrace{\uparrow_0^1 (\lambda. 12)}]}$$

$$\cancel{= \lambda.\ \uparrow_1^1 (1 2)}$$

$$= \lambda.\ 1$$

$$\lambda.\ \overbrace{(\lambda. \lambda. 1\ 0)\ (\lambda. 0\ 1)}$$

$$\to_\beta \lambda.\ \uparrow_0^{-1} (\ \lambda. 1 0) [0 / \underbrace{\uparrow_0^1 (\lambda. 0\ 1)}_{= \lambda. 0 2}]$$

$$= \lambda.\ \uparrow_0^{-1} (\ \lambda. (1 0) [1 / \underbrace{\uparrow_0^1 (\lambda. 0 2)}_{= \lambda. 0 3}]$$

$$= \lambda. \uparrow_0^{-1} (\lambda. (\lambda. 0\ 3)\ 0) =$$

$$= \lambda.\ \lambda.\ (\lambda.\ \uparrow_2^{-1} (0 3))\ \uparrow_1^{-1} (0))$$

$$= \lambda. \lambda.\ \underbrace{(\lambda.\ 0\ 2)\ 0}$$

$$\to_\beta \lambda.\ \lambda.\ \uparrow_0^{-1} ((0\ 2) [0 / \underbrace{\uparrow_0^1 (0)}_{= 1}]$$

$= \lambda.\lambda.\uparrow_0^{-1}(1\ 2)$

$= \lambda.\lambda.0\ 1 \sim \lambda x.\lambda z.z x$

$\blacksquare$

Зам. Фиолетовым отмечены дополнения и исправления.

Кодирование

$T = \lambda x.\lambda y.x$

$F = \lambda x.\lambda y.y$

$[u, v] \xleftarrow{\text{pair}} = \underline{\lambda z.(z\ u)\ v}$

$[u.v]\ T \to_\beta u$

$[u.v]\ F \to_\beta v$

$d_0 = \lambda x.x$

$d_{n+1} = [F, d_n]$

$0 = \varnothing$

$1 = \{\varnothing\}$

$2 = \{\varnothing, \{\varnothing\}\}$

$\ldots$

Th $f : \mathbb{N}^k \to \mathbb{N}$ частичные вычислимы

определен. ф-ии

$\Longleftrightarrow$

f в виде λ-терма $f(n_1 \ldots n_k) = m$

тогда $((F\ d_{n_1}) \ldots d_{n_k}) \to_\beta d_m$

$\to_\beta^*$

① ПРФ

$f: N^k \to N$

1. базисные

$Z = \lambda x. d_0$ $\qquad z(x) = 0$

$S = \lambda x. [F, x]$ $\qquad S(x) = x+1$

$2. S: N \to N$

zero

succ

$I_k^n = \lambda x_1 \dots \lambda x_n . x_k$ $\qquad i_k^n(x_1, \dots, x_n) = x_k$

$i_k^n : N^n \to N$

2. композиция

$g: N^m \to N, \quad f_1, \dots f_m : N^n \to N$

$$h(\underbrace{x_1, \dots, x_n}_{\leftrightharpoons \vec{x}}) = g(f_1(\vec{x}), \dots, f_m(\vec{x}))$$

3. прим. рекурсия

g, h – ПРФ

$$\begin{cases} f(0, \vec{x}) = g(\vec{x}) \\ f(n+1, \vec{x}) = h(f(n, \vec{x}), n, \vec{x}) \end{cases}$$

$G \sim g \qquad F_1 \dots F_m \sim f_1 \dots f_m$

$$H = \lambda x_1 \dots \lambda x_n . G\ (F_1\ x_1 \dots x_n) \dots (F_m\ x_1 \dots x_n)$$

$F = \lambda y . \lambda x_1 \dots \lambda x_k .$ <u>if</u> zero y

<u>then</u> $G\ x_1 \dots x_k$

<u>else</u> $H\ (F (y-1)\ \vec{x})\ (y-1)\ \vec{x}$

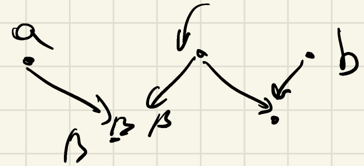
1. if-then-else
2. $y-1$
3. zero
4. рекурсия

Th $\forall F \; \exists V : \; V =_\beta F V$

Proof

$V = (\lambda x. F(xx))(\lambda x. F(xx))$

$a =_\beta b$



Th 2

$\exists Y : \forall F. \; Y F \twoheadrightarrow_\beta F(Y F)$

Proof

1) $Y = \lambda f. (\lambda x. \; f(xx))(\lambda x. f(xx))$

2) $\Theta = A A \; , \; A = \lambda x. \lambda y. y (x x y)$

3) ...

$V = \underbrace{(\lambda x. F(xx))}(\lambda x. F(xx)) =$

$\twoheadrightarrow_\beta F((\lambda x. F(xx))(\lambda x. F(xx)))$

$= F V$

$(\lambda x. e_1) e_2 \rightarrow_\beta e_1[x/e_2]$

Th $\forall M\ \exists F : F =_{\beta} M[f/F]$

Proof $F = \mathbb{Y}(\lambda f. M)$

fresh

$$F = \mathbb{Y}(\lambda f. \lambda y. \lambda \vec{x}.\ \underline{\text{if}}\ \text{zero}\ y$$
$$\underline{\text{then}}\ G\ \vec{x}$$
$$\underline{\text{else}}\ H\ (f\ (\text{pred}\ y)\ \vec{x})\ (\text{pred}\ y)\ \vec{x})$$

$$\cancel{F = \mathbb{Y}(\lambda f. M) = (\lambda g. (\lambda x. g(xx))(\lambda x. g(xx)))(\lambda f. M)}$$
$$\cancel{= (\lambda x. (\lambda f. M)(xx))(\lambda x. (\lambda f. M)(xx))}$$
$$\cancel{= (\lambda x. M[f/(xx)])(\lambda x. M[f/(xx)])}$$
$$\cancel{= M[f/(xx)][x/\lambda x. M[f/(xx)]]}$$

$$F \equiv \mathbb{Y}(\lambda f. M) \overset{2}{=} (\lambda f. M)(\mathbb{Y}(\lambda f. M))$$

$\mathbb{Y}F \to_{\beta} F(\mathbb{Y}F)$

$$\to_{\beta} M[f/\underbrace{\mathbb{Y}(\lambda f. M)}_{F}]$$
$$= M[f/F]$$

$$\underline{\text{if}}\ B\ \underline{\text{then}}\ u\ \underline{\text{else}}\ v \overset{\text{def}}{\equiv} B\,U\,V$$

$$T\,U\,V = U$$
$$F\,U\,V = V$$

$$\text{zero} \equiv \lambda x.\, x\, T$$

$$\lambda y.y = 0$$

$$(\lambda y.y)T = T$$

$$\{F, \ldots [F, [F, d_0]]\}$$

$$F$$

$$\text{pred} = \lambda x.\, x\, F$$

$$\lambda s.\lambda z.\, z$$

$$\lambda s.\lambda z.\, s z$$

$$\lambda s.\lambda z.\, s^k(z) - d_k$$

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zacs. bez.

$$\mu z.\, g(z, \vec{x}) = \min\{z \mid g(z, \vec{x}) = 0\}$$

$$\mu z.\, g(z, \vec{x}) = \begin{cases} m, & \underline{\text{ecли}}\ g(m, \vec{x}) = 0 \\ & \text{и min m} \\ \text{не опр} & \end{cases}$$

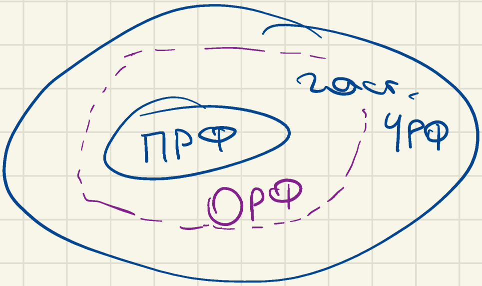
Th
$\forall f : \mathbb{N}^n \to \mathbb{N}$ (част.)
$\exists e \in \mathbb{N}$

U, T – прим.-рек. ф-ии

$$f(\vec{x}) = U(\mu z . T(e, \vec{x}, z))$$

прим прим

$ЧРФ = ПРФ + \mu$



$ОРФ = ЧРФ$ тотальные (!) только про них и говорим

(!) ели μ выразимо в Λ

$\rightarrow$ $h(t, \vec{x}) \stackrel{def}{=} \begin{cases} t, & \text{если } g(t, \vec{x}) = 0 \\ h(t+1, \vec{x}) & \text{иначе} \end{cases}$

$f(\vec{x}) \stackrel{def}{=} h(0, \vec{x})$

тогда

$$H \stackrel{def}{=} Y(\lambda h. \lambda t. \lambda \vec{x}. \text{ if } Zero\,(G\,t\,\vec{x}) \text{ then } t \text{ else } h\,(s\,t)\,\vec{x}$$

Осталось (!)$^{\text{ть}}$, что терм „правильно" редуцируется

$\exists\, h(t, x) = p$, т.е. $p \geq t$ и первое такое, что g обращается в 0

Тогда по индукции верно для p.

ex $p = t + 1$ (ИП)

$$H\, d_t\, d_x =_\beta \text{if } \underbrace{\text{zero}\,(G\, d_t\, d_x)}_{F \text{ по ИП}} \text{ then } d_t \text{ else } H\,(S\, d_t)\, d_x$$

$$=_\beta \text{if } \underbrace{\text{zero}\,(G\, d_{t+1}\, d_x)}_{T \text{ по ИП}} \text{ then } d_{t+1} \text{ else } H\,(S\, d_{t+1})\, d_x$$

$$=_\beta d_{t+1}$$

Чтд

Зам: индукция работает всходствие всюдуопределенности ф-ии.

Пример Factorial

$$g = \lambda fct.\ \lambda n.\ \underline{if}\ zero\ n\ \underline{then}\ d_1\ \underline{else}\ times\ n\ (fct\ (pred\ n))$$

$$factorial = fix\ g$$

где $fix = \lambda f.\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))$

▶ $factorial\ d_3 = fix\ g\ d_3$

$$\to_\beta (\lambda x.\ g\ (\lambda y.\ x\ x\ y))\ \underbrace{(\lambda x.\ g\ (\lambda y.\ x\ x\ y))}_{\leftrightharpoons h}\ d_3$$

$$= h\ h\ d_3$$

$$\to_\beta g\ \underbrace{(\lambda y.\ h\ h\ y)}_{\leftrightharpoons fct}\ d_3$$

$$\to_\beta (\lambda n.\ \underline{if}\ zero\ n\ \underline{then}\ d_1\ \underline{else}\ times\ n\ (fct\ (pred\ n)))\ d_3$$

$$\twoheadrightarrow_\beta times\ d_3\ (fct\ (pred\ d_3))$$

$$\twoheadrightarrow_\beta times\ d_3\ (fct\ d_2)$$

$$= times\ d_3\ ((\lambda y.\ h\ h\ y)\ d_2)$$

$$\to_\beta times\ d_3\ (h\ h\ d_2)$$

аналогично

$$\twoheadrightarrow_\beta times\ d_3\ (times\ d_2\ (h\ h\ d_1))$$

аналогично

$$\twoheadrightarrow_\beta times\ d_3\ (times\ d_2\ (times\ d_1\ (h\ h\ d_0)))$$

$$\twoheadrightarrow_\beta$$

& $h\,h\,d_0$

$$= (\lambda x.\, g\,(\lambda y.\, x\,x\,y))\,(\lambda x.\, g\,(\lambda y.\, x\,x\,y))\,d_0$$

$$\twoheadrightarrow_\beta g\,\underbrace{(\lambda y.\, h\,h\,y)}_{\leftrightharpoons fct'}\,d_0$$

$$\twoheadrightarrow_\beta (\lambda n.\ \underline{\text{if}}\ \text{zero}\ n\ \text{then}\ d_1\ \underline{\text{else}}\ \text{times}\ n\ (fct'\ (\text{pred}\ n)))\,d_0$$

$$\twoheadrightarrow_\beta \underline{\text{if}}\ \text{zero}\ d_0\ \underline{\text{then}}\ d_1\ \underline{\text{else}}\ \ldots$$

$$\twoheadrightarrow_\beta d_1$$

$$\twoheadrightarrow_\beta \text{times}\ d_3\,(\text{times}\ d_2\,(\text{times}\ d_1\,(\text{times}\ d_1)))$$

$$\twoheadrightarrow_\beta d_6$$

при правильном определении ф-ии умножения times [упражнение для самост. работы!]

$\blacksquare$