

On Lambda Calculus

Reductions

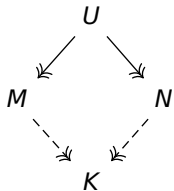
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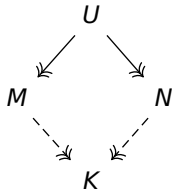
## Theorem [confluence]

$$\begin{aligned} U \rightarrow_{\beta} M, U \rightarrow_{\beta} N \\ \Rightarrow \\ \exists K : M \rightarrow_{\beta} K, N \rightarrow_{\beta} K \end{aligned}$$



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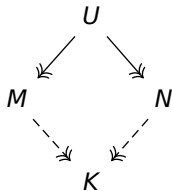


## Ideally: Dimond property

$$U \rightarrow M, U \rightarrow N \Rightarrow \exists K : M \rightarrow K, N \rightarrow K$$

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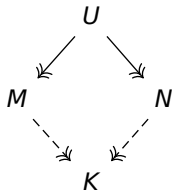
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Doesn't hold for  $\beta$ : let  $v \rightarrow_{\beta} v'$

$$\begin{array}{ccc}
 (\lambda xy.yxx)x & \rightarrow & \lambda y.yvv \\
 \downarrow & & \downarrow \\
 & & \lambda y.yv'v \\
 \downarrow & & \downarrow \\
 (\lambda xy.yxx)v' & \rightarrow & \lambda y.yv'v'
 \end{array}$$

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## Weak Church Rosser

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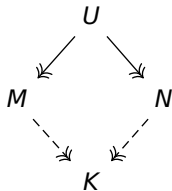
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 (\lambda xy.yxx)v' & \longrightarrow & \lambda y.yv'v'
 \end{array}$$

NB:  $WCR \not\Rightarrow CR$  in general:

## Newman's lemma

$$WCR + SN \Rightarrow CR$$

# Normal Forms

Reduction under $\lambda$	Yes	No
Reduction of arguments	Yes	No
Yes	<b>Strong NF</b> $N ::= \lambda x.N \mid x \ N_1 \dots N_n$	<b>Weak NF</b> $W ::= \lambda x.e \mid x \ W_1 \dots W_n$
No	<b>Head NF</b> $H ::= \lambda x.H \mid x \ e_1 \dots e_n$	<b>Weak head NF</b> $E ::= \lambda x.e \mid x \ e_1 \dots e_n$

Leftmost outermost weak

$$\frac{x \xrightarrow{bn} x \quad e_1 \xrightarrow{bn} \lambda x.e \quad e[x/e_2] \xrightarrow{bn} e'}{e_1 e_2 \xrightarrow{bn} e'}$$

$$\frac{\lambda x.e \xrightarrow{bn} \lambda x.e \quad e_1 \xrightarrow{bn} e'_1 \not\equiv \lambda x.e}{e_1 e_2 \xrightarrow{bn} e'_1 e_2}$$

- WHNF
- Repeated computations



Leftmost outermost strong

$$x \xrightarrow{no} x$$

$$\frac{e_1 \xrightarrow{bn} \lambda x.e \quad e[x/e_2] \xrightarrow{no} e'}{e_1 e_2 \xrightarrow{no} e'}$$

$$\frac{e \xrightarrow{no} e'}{\lambda x.e \xrightarrow{no} \lambda x.e'}$$

$$\frac{e_1 \xrightarrow{bn} e'_1 \neq \lambda x.e \quad e'_1 \xrightarrow{no} e''_1 \quad e_2 \xrightarrow{no} e'_2}{e_1 e_2 \xrightarrow{no} e''_1 e'_2}$$

- NF
- Is normalizing
- Repeated computations

Leftmost innermost weak

$$\begin{array}{c}
 x \xrightarrow{bv} x \\
 \hline
 e_1 \xrightarrow{bv} \lambda x.e \quad e_2 \xrightarrow{bv} e_2' \quad e[x/e_2'] \xrightarrow{bv} e' \\
 \hline
 e_1 e_2 \xrightarrow{bv} e'
 \end{array}
 \qquad
 \begin{array}{c}
 \lambda x.e \xrightarrow{bv} \lambda x.e \\
 \hline
 e_1 \xrightarrow{bv} e_1' \neq \lambda x.e \quad e_2 \xrightarrow{bv} e_2' \\
 \hline
 e_1 e_2 \xrightarrow{bv} e_1' e_2'
 \end{array}$$

- WNF
- May diverge even if WHF exists

Leftmost innermost strong

$$x \xrightarrow{ao} x$$

$$\frac{e_1 \xrightarrow{an} \lambda x.e \quad e_2 \xrightarrow{ao} e'_2 \quad e[x/e'_2] \xrightarrow{ao} e'}{e_1 e_2 \xrightarrow{ao} e'}$$

$$\frac{e \xrightarrow{ao} e'}{\lambda x.e \xrightarrow{ao} \lambda x.e'}$$

$$\frac{e_1 \xrightarrow{ao} e'_1 \neq \lambda x.e \quad e_2 \xrightarrow{no} e'_2}{e_1 e_2 \xrightarrow{ao} e'_1 e'_2}$$

- NF
- May diverge even if NF exists



# Hybrid applicative order

$$\begin{array}{c}
 x \xrightarrow{ha} x \\
 \\
 \frac{e_1 \xrightarrow{bv} \lambda x.e \quad e_2 \xrightarrow{ha} e'_2 \quad e[x/e'_2] \xrightarrow{ha} e'}{e_1 e_2 \xrightarrow{ha} e'} \quad \frac{\frac{e \xrightarrow{ha} e'}{\lambda x.e \xrightarrow{ha} \lambda x.e'}}{e_1 \xrightarrow{bv} e'_1 \neq \lambda x.e \quad e'_1 \xrightarrow{ha} e''_1 \quad e_2 \xrightarrow{ha} e'_2}}{e_1 e_2 \xrightarrow{ha} e''_1 e'_2}
 \end{array}$$

- NF
- Still may diverge even if NF exists
- Normalizes more than AO
- Works with  $Y_v$

$$\begin{array}{c}
 x \xrightarrow{he} x \\
 \\
 \frac{e_1 \xrightarrow{he} \lambda x.e \quad e[x/e_2] \xrightarrow{he} e'}{e_1 e_2 \xrightarrow{he} e'}
 \end{array}
 \qquad
 \frac{e \xrightarrow{he} e'}{\lambda x.e \xrightarrow{he} \lambda x.e'}$$

$$\frac{e_1 \xrightarrow{he} e'_1 \neq \lambda x.e}{e_1 e_2 \xrightarrow{ha} e'_1 e_2}$$

- HNF
- Head reduction = spine head + bn (as in no)

# (Leftmost) Head Reduction

Any term  $T$  can be written uniquely as  $T = \lambda x_1 \dots \lambda x_n. U V_1 \dots V_n$  where

$$U = \begin{cases} y & \text{HNF} \\ (\lambda x. e)V & \text{Head redex} \end{cases}$$

Recursive application to arguments is normalizing

**HNF is not unique!**

$y ((\lambda x. x) z) \rightarrow_{\beta} y z$

**Principal** HNF — the one obtained by HE if terminates

**Böhm tree**

$$BT(M) ::= \begin{cases} \begin{array}{c} \omega \\ \vec{\lambda x}. y \\ \swarrow \quad \downarrow \quad \searrow \\ BT(V_1) \quad \dots \quad BT(V_n) \end{array} & \text{if } M \text{ has no HNF} \\ phnf(M) = \lambda \vec{x}. y V_1 \dots V_n & \end{cases}$$

# (Leftmost) Head Reduction

Term  $T$  has NF iff  $\exists$  finite  $BT(T)$

Term  $T$  has no NF iff either

>  $BT(T)$  is finite but the next level evaluation diverges

$$BT(\lambda x \omega(\lambda x)) = \begin{array}{c} x \\ / \backslash \\ \omega \quad x \end{array}$$

>  $BT(T)$  is infinite

$$BT(\lambda f.(\lambda x.f(xx))(\lambda x.f(xx))) = \begin{array}{c} \lambda f.f \\ \vdots \\ f \\ | \\ f \\ \dots \end{array}$$