On Lambda Calculus

Reductions

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Church–Rosser

Theorem [confluence]

$$U \twoheadrightarrow_\beta M,\ U \twoheadrightarrow_\beta N$$
$$\Rightarrow$$
$$\exists K :\ M \twoheadrightarrow_\beta K,\ N \twoheadrightarrow_\beta K$$

$$\begin{tikzcd}
 & U \arrow[dl, two heads] \arrow[dr, two heads] & \\
M \arrow[dr, two heads, dashed] & & N \arrow[dl, two heads, dashed] \\
 & K &
\end{tikzcd}$$

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Ideally: Dimond property

$$U \to M,\ U \to N \Rightarrow \exists K:\ M \to K,\ N \to K$$

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Doesn't hold for β: let $v \to_\beta v'$

$$\begin{tikzcd}
(\lambda xy.yxx)x \arrow[r] \arrow[dd] & \lambda y.yvv \arrow[d] \\
 & \lambda y.yv'v \arrow[d] \\
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Weak Church Rosser

$$U \to M,\ U \to N \Rightarrow \exists K:\ M \twoheadrightarrow K,\ N \twoheadrightarrow K$$

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Church–Rosser

Theorem [confluence]

$$U \twoheadrightarrow_\beta M,\ U \twoheadrightarrow_\beta N$$
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 & U \arrow[dl, ->>] \arrow[dr, ->>] & \\
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 & K &
\end{tikzcd}$$

Weak Church Rosser

$$U \to M,\ U \to N \Rightarrow \exists K:\ M \twoheadrightarrow K,\ N \twoheadrightarrow K$$

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\end{tikzcd}$$

NB: $WCR \not\Rightarrow CR$ in general:

Newman's lemma

$$WCR + SN \Rightarrow CR$$

Normal Forms

Reduction of agruments \ Reduction under λ	Yes	No
Yes	Strong NF $N ::= \lambda x.N \mid x\ N_1 \dots\ N_n$	Weak NF $W ::= \lambda x.e \mid x\ W_1 \dots\ W_n$
No	Head NF $H ::= \lambda x.H \mid x\ e_1 \dots e_n$	Weak head NF $E ::= \lambda x.e \mid x\ e_1 \dots e_n$

Leftmost outermost weak

$$x \xrightarrow{bn} x \qquad\qquad \lambda x.e \xrightarrow{bn} \lambda x.e$$

$$\frac{e_1 \xrightarrow{bn} \lambda x.e \quad e[x/e_2] \xrightarrow{bn} e'}{e_1\ e_2 \xrightarrow{bn} e'} \qquad\qquad \frac{e_1 \xrightarrow{bn} e_1' \not\equiv \lambda x.e}{e_1\ e_2 \xrightarrow{bn} e_1' e_2}$$

- WHNF
- Repeated computations

Leftmost outermost strong

$$x \overset{no}{\rightarrow} x \qquad \frac{e \overset{no}{\rightarrow} e'}{\lambda x.e \overset{no}{\rightarrow} \lambda x.e'}$$

$$\frac{e_1 \overset{bn}{\rightarrow} \lambda x.e \quad e[x/e_2] \overset{no}{\rightarrow} e'}{e_1\ e_2 \overset{no}{\rightarrow} e'} \qquad \frac{e_1 \overset{bn}{\rightarrow} e_1' \not\equiv \lambda x.e \quad e_1' \overset{no}{\rightarrow} e_1'' \quad e_2 \overset{no}{\rightarrow} e_2'}{e_1\ e_2 \overset{no}{\rightarrow} e_1'' e_2'}$$

- NF
- Is normalizing
- Repeated computations

Leftmost innermost weak

$$x \xrightarrow{bv} x \qquad \lambda x.e \xrightarrow{bv} \lambda x.e$$

$$\frac{e_1 \xrightarrow{bv} \lambda x.e \quad e_2 \xrightarrow{bv} e2' \quad e[x/e_2'] \xrightarrow{bv} e'}{e_1\ e_2 \xrightarrow{bv} e'} \qquad \frac{e_1 \xrightarrow{bv} e_1' \not\equiv \lambda x.e \quad e_2 \xrightarrow{bv} e_2'}{e_1\ e_2 \xrightarrow{bv} e_1' e_2'}$$

- WNF
- May diverge even if WHF exists

Applicative order

Leftmost innermost strong

$$x \xrightarrow{ao} x \qquad \frac{e \xrightarrow{ao} e'}{\lambda x.e \xrightarrow{ao} \lambda x.e'}$$

$$\frac{e_1 \xrightarrow{an} \lambda x.e \quad e_2 \xrightarrow{ao} e_2' \quad e[x/e_2'] \xrightarrow{ao} e'}{e_1\ e_2 \xrightarrow{ao} e'} \qquad \frac{e_1 \xrightarrow{ao} e_1' \not\equiv \lambda x.e \quad e_2 \xrightarrow{no} e_2'}{e_1\ e_2 \xrightarrow{ao} e_1' e_2'}$$

- NF
- May diverge even if NF exists

Hybrid applicative order

$$x \xrightarrow{ha} x \qquad \frac{e \xrightarrow{ha} e'}{\lambda x.e \xrightarrow{ha} \lambda x.e'}$$

$$\frac{e_1 \xrightarrow{bv} \lambda x.e \quad e_2 \xrightarrow{ha} e_2' \quad e[x/e_2'] \xrightarrow{ha} e'}{e_1\ e_2 \xrightarrow{ha} e'} \qquad \frac{e_1 \xrightarrow{bv} e_1' \not\equiv \lambda x.e \quad e_1' \xrightarrow{ha} e_1'' \quad e_2 \xrightarrow{ha} e_2'}{e_1\ e_2 \xrightarrow{ha} e_1'' e_2'}$$

- NF
- Still may diverge even if NF exists
- Normalizes more than AO
- Works with Y_v

Spine Head Reduction

$$x \xrightarrow{he} x \qquad \frac{e \xrightarrow{he} e'}{\lambda x.e \xrightarrow{he} \lambda x.e'}$$

$$\frac{e_1 \xrightarrow{he} \lambda x.e \quad e[x/e_2] \xrightarrow{he} e'}{e_1\ e_2 \xrightarrow{he} e'} \qquad \frac{e_1 \xrightarrow{he} e'_1 \not\equiv \lambda x.e}{e_1\ e_2 \xrightarrow{ha} e'_1 e_2}$$

- HNF
- Head reduction = spine head + bn (as in no)

Any term T can be written uniquely as $T = \lambda.x_1 \dots \lambda x_n.U\ V_1 \dots V_n$ where

$$U = \left[\begin{array}{ll} y & \text{HNF} \\ (\lambda x.e)V & \text{Head redex} \end{array}\right.$$

Recurcive application ro arguments is normalizing

HNF is not unique!

$y\ ((\lambda x.\ x)\ z) \to_\beta y\ z$

Principal HNF — the one obtained by HE if terminates

Bëhm tree

$$BT(M) ::= \begin{cases} \omega & \text{if M has no HNF} \\ \begin{array}{c} \lambda\vec{x}.y \\ BT(V_1)\ \dots\ BT(V_n) \end{array} & phnf(M) = \lambda \xrightarrow{x} .\ y\ V_1 \dots V_n \end{cases}$$

Term T has NF <u>iff</u> $\exists$ finite $BT(T)$

Term T has no NF <u>iff</u> either

- BT(T) is finite but the next level evaluation diverges

$$BT(Ix\omega(Ix)) = \begin{array}{c} x \\ /\backslash \\ \omega\ x \end{array}$$

- BT(T) is infinite

$$BT(\lambda f.(\lambda x.f(xx))(\lambda x.f(xx))) = \begin{array}{c} \lambda f.f \\ \vdots \\ f \\ | \\ f \\ \ldots \end{array}$$